Gauss's Law, Gauge-Invariant States, and Spin and Statistics In Abelian Chern-Simons Theories.

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Abstract

We discuss topologically massive QED — the Abelian gauge theory in which (2+1)-dimensional QED with a Chern-Simons term is minimally coupled to a spinor field. We quantize the theory in covariant gauges, and construct a class of unitary transformations that enable us to embed the theory in a Fock space of states that implement Gauss's law. We show that when electron (and positron) creation and annihilation operators represent gauge-invariant charged particles that are surrounded by the electric and magnetic fields required by Gauss's law, the unitarity of the theory is manifest, and charged particles interact with photons and with each other through nonlocal potentials. These potentials include a Hopf-like interaction, and a planar analog of the Coulomb interaction. The gauge-invariant charged particle excitations that implement Gauss's law obey the identical anticommutation rules as do the original gauge-dependent ones. Rotational phases, commonly identified as planar 'spin', are arbitrary, however.

I. INTRODUCTION

In this report we will summarize and extend our earlier work on Maxwell-Chern-Simons (MCS) and Chern-Simons (CS) theory, [1–3] and discuss the properties of the particle excitations produced when relativistic fields — spinors, in this case — are minimally coupled to Abelian (2+1)-dimensional gauge theories that include Chern-Simons terms. This work will also illustrate canonical quantization as a method for exploring the properties of the particle excitations of gauge fields and of the fields coupled to them. In this, as in earlier work — we will use MCS theory in covariant gauges as an illustrative example here — we will construct gauge-invariant excitations of the charged spinor fields that satisfy Gauss's law and the appropriate gauge condition. We will also demonstrate that these charged particle excitations — we will refer to them as 'electrons' and 'positrons,' even though they appear in a (2+1)-dimensional space — interact through an effective Hamiltonian that consists of a

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vertex interaction at which two charged spinor lines meet a topologically massive 'photon,' and through nonlocal interactions, one of which has a long-range tail with an Aharonov-Bohm-like structure (the so-called Hopf interaction). [4] One of our main objectives is to explore the rotational and statistical properties of the gauge-invariant particle states that obey Gauss's law.

II. DYNAMICAL CONSIDERATIONS

The Lagrangian for MCS theory minimally coupled to a spinor field is

$$\mathcal{L}_{\text{MCS}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} m\epsilon_{\mu\nu\lambda} F^{\mu\nu} A^\lambda - j_\mu A^\mu - \bar{\psi}(M - i\gamma_\mu \partial^\mu)\psi + \mathcal{L}_{\text{gf}} \tag{1}$$

where $j^\mu = e\bar{\psi}\gamma^\mu\psi$ and m is the Chern-Simons coefficient which, in Abelian theories, can have arbitrary real values. $\mathcal{L}_{\text{gf}}$ represents the gauge-fixing Lagrangian; in our work, we have used forms that correspond to the following gauges:

$$\mathcal{L}_{\text{gf}} = \begin{cases} -G\partial_\mu A^\mu - \frac{1}{2}(1-\gamma)G^2 & \text{covariant} \\ \quad \gamma = 0 \rightarrow \text{Feynman} \\ \quad \gamma = 1 \rightarrow \text{Landau} \\ -G\partial_0 A_0 & \text{temporal} \\ -G\nabla \cdot \mathbf{A} & \text{Coulomb.} \end{cases} \tag{2}$$

In covariant gauges, there are trivial constraints only, so that the canonical quantization of this model can be carried out without requiring any special procedures for implementing primary constraints. All components of the gauge field, A_0 as well as A_l, have canonically conjugate momenta given by

$$\Pi_l = F_{0l} + \frac{1}{2} m\epsilon_{ln} A_n \quad (F_{0l} = -E_l) \text{ and } \Pi_0 = -G, \tag{3}$$

and they are subject to the standard equal-time commutation or anticommutation rules (ETCR)

$$[A_0(\mathbf{x}), G(\mathbf{y})] = -i\delta(\mathbf{x} - \mathbf{y}), \tag{4}$$

$$[A_l(\mathbf{x}), \Pi_n(\mathbf{y})] = i\delta_{ln}\delta(\mathbf{x} - \mathbf{y}), \tag{5}$$

$$\text{and} \quad \{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}). \tag{6}$$

The Hamiltonian for covariant gauges is given by $H = H_0 + H_{\text{I}}$, where

$$H_0 = \int d\mathbf{x} \left[\frac{1}{2} \Pi_l \Pi_l + \frac{1}{4} F_{ln} F_{ln} + G\partial_l A_l - \frac{1}{2}(1-\gamma)G^2 + \frac{1}{8} m^2 A_l A_l \right.$$
$$\left. + A_0 \left(\partial_l \Pi_l - \frac{1}{4} m\epsilon_{ln} F_{ln} + j_0 \right) - j_l A_l + \frac{1}{2} m\epsilon_{ln} A_l \Pi_n \right] + H_{e\bar{e}} \tag{7}$$

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and

$$H_{\text{I}} = \int d\mathbf{x} \ (j_0 A_0 - j_l A_l) . \tag{8}$$

In order to associate these fields with particle excitations, we expand the operator-valued fields in terms of creation and annihilation operators for particle excitations. These excitations include two excitation modes of the spinor field $\psi(\mathbf{x})$ — 'electrons' with spin $1/2$ and 'positrons' with spin $-1/2$ — and a single propagating topologically massive mode of the gauge fields, originally described by Deser, Jackiw and Templeton. [5] Annihilation operators for electrons and positrons of momentum $\mathbf{k}$ are represented as e_k and $\bar{e}_\mathbf{k}$ respectively, and the corresponding creation operators are $e_\mathbf{k}^\dagger$ and $\bar{e}_\mathbf{k}^\dagger$. Annihilation and creation operators for the topologically massive photons are represented as $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ respectively. In the covariant gauge, and in any gauge with only trivial primary constraints, 'ghost' modes are also necessary to represent the ETCR for all components of the gauge field. There are two different varieties of ghost modes, 'Q' and 'R' ghosts, whose annihilation and creation operators, respectively, are $a_Q(\mathbf{k}), a_R(\mathbf{k})$, and $a_Q^\star(\mathbf{k}), a_R^\star(\mathbf{k})$. Since the one-particle ghost states must have zero norm, $a_Q(\mathbf{k})$ and its conjugate creation operator $a_Q^\star(\mathbf{k})$ must commute, as must $a_R(\mathbf{k})$ and $a_R^\star(\mathbf{k})$. But since the ghost modes must support the ETCR of the gauge fields, not *all* the ghost operators can commute. The commutation rules for ghost operators are

$$[a_Q(\mathbf{k}), a_R^\star(\mathbf{q})] = [a_R(\mathbf{k}), a_Q^\star(\mathbf{q})] = \delta_{\mathbf{kq}}, \tag{9}$$

and the unit operator in the one particle ghost (OPG) sector is

$$1_{\text{opg}} = \sum_{\mathbf{k}} \left[a_Q^\star(\mathbf{k})|0\rangle\langle 0|a_R(\mathbf{k}) + a_R^\star(\mathbf{k})|0\rangle\langle 0|a_Q(\mathbf{k}) \right] . \tag{10}$$

The excitation operators we have described are used to represent the spinor and gauge fields in such a way that they implement the space-time commutation rules, Eqs. (4)–(5). They must also result in a Hamiltonian — the 'free field' Hamiltonian, in the first instance — that produces only one kind of ghost in the course of time evolution. When only a single variety of ghost coexists with the propagating observable states (we will always select the 'Q' ghost to fulfill this role), then the states with ghost content are purely zero-norm states, and they do not drain any probability from the part of the Hilbert space that represents physically observable configurations. When the two ghosts coexist, the states with mixed ghost content are not zero-norm states, and threaten the unitarity of the theory by draining probability from the space of propagating particles to the 'ghost' part of the Hilbert space.

The representation of the gauge fields in terms of the particle and ghost annihilation and creation operators is given in Ref. [3], and lack of space prevents its repetition here. But we will specify the 'free' Hamiltonian, H_0, which is given by

$$H_0 = \sum_{\mathbf{k}} \frac{\omega_k}{2} \left[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) \right] + \sum_{\mathbf{k}} k \left[a_Q^\star(\mathbf{k})a_R(\mathbf{k}) + a_Q(\mathbf{k})a_R^\star(\mathbf{k}) \right]$$

$$- (1 - \gamma) \sum_{\mathbf{k}} \frac{64k^4}{m^3} a_Q^\star(\mathbf{k})a_Q(\mathbf{k}) + \int d\mathbf{x} \ \psi^\dagger(\gamma_0 M - i\gamma_0\gamma_l\partial_l)\psi. \tag{11}$$

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The interaction Hamiltonian, H_I, in the same representation, is also given in Ref. [3]. In contrast to H_0, H_I has a profusion of *both* varieties of ghost creation and annihilation operators, so that we must take further steps to protect the unitarity of the theory. We also have to respond to the fact that $e_{\mathbf{k}}$ and $\bar{e}_{\mathbf{k}}$ annihilate, and $e_{\mathbf{k}}^{\dagger}$ and $\bar{e}_{\mathbf{k}}^{\dagger}$ create, electrons and positrons that are entirely 'free'—*i.e.* unaccompanied by any electric and magnetic fields—so that they fail to implement Gauss's law.

III. IMPLEMENTING GAUSS'S LAW

Gauss's law for MCS theory is expressed by

$$\partial_l F_{0l} - \frac{1}{2} m \epsilon_{ln} F_{ln} + j_0 = 0 \tag{12}$$

and the 'Gauss's law operator' $\mathcal{G}(\mathbf{x})$ is given by

$$\mathcal{G} = \partial_l F_{0l} - \frac{1}{2} m \epsilon_{ln} F_{ln} + j_0 = \partial_l \Pi_l - \frac{1}{4} m \epsilon_{ln} F_{ln} + j_0. \tag{13}$$

$\mathcal{G}(\mathbf{x})$ can be represented as

$$\mathcal{G}(\mathbf{x}) = \sum_{\mathbf{k}} \frac{8k^3}{m^{3/2}} \left[\Omega(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \Omega^{\star}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{14}$$

where

$$\Omega(\mathbf{k}) = a_Q(\mathbf{k}) + \frac{m^{3/2}}{16k^3} j_0(\mathbf{k}) \tag{15}$$

with

$$j_0(\mathbf{k}) = \int d\mathbf{x}\, j_0(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{16}$$

$\Omega(\mathbf{k})$ can be used to impose Gauss's law on a set of states $\{|\nu\rangle\}$ since it obeys

$$[H, \Omega(\mathbf{k})] = -k\Omega(\mathbf{k}) \tag{17}$$

so that $\Omega(\mathbf{k})$ is operator-valued, but has a *c*-number time dependence in the Heisenberg picture,

$$\Omega(\mathbf{k}, t) = e^{iHt} \Omega(\mathbf{k}) e^{-iHt} = \Omega(\mathbf{k}) e^{-ikt}. \tag{18}$$

Hence,

$$\Omega(\mathbf{k})|\nu\rangle = 0, \tag{19}$$

if imposed at one time $(t = 0)$ remains in force forever!

We now observe that, because the gauge group of this theory is Abelian, $\Omega(\mathbf{k})$ is unitarily equivalent to $a_Q(\mathbf{k})$, so that there are unitary operators, U,—many, in fact—for which

$$U^{-1}\Omega(\mathbf{k})U = a_Q(\mathbf{k}). \tag{20}$$

Examples of U operators are given in Ref. [3], but, because of limited space, will not be repeated here. The important point to emphasize is that there are two alternate ways of using U operators to construct states that implement Gauss's law: We can use U to construct states $|\nu\rangle$ that satisfy $\Omega(\mathbf{k})|\nu\rangle = 0$, for which H is the generator of time-evolutions. Alternatively — and this is the option we will use here — we can transpose the *entire formulation* to a new representation using

$$\hat{\mathcal{P}} = U^{-1}\mathcal{P}U \quad (\mathcal{P} \text{ represents any operator}) \tag{21}$$

to transform all operators to a new representation. Then $U^{-1}\Omega(\mathbf{k})U = \hat{\Omega}(\mathbf{k}) = a_Q(\mathbf{k})$,

$$U^{-1}HU = \hat{H} \tag{22}$$

and similarly for all gauge field (and spinor) operators, as well as all dynamical variables. This similarity transformation preserves all algebraic relationships among transformed operators. The unitary transforms of the states, $|\nu\rangle$, that satisfy Eq. (19) are given by

$$\Omega(\mathbf{k})|\nu\rangle = 0 \quad \Rightarrow \quad \hat{\Omega}(\mathbf{k})|n\rangle = a_Q(\mathbf{k})|n\rangle = 0. \tag{23}$$

The states, $|n\rangle$, constitute the Fock space built on the perturbative vacuum $|0\rangle$ annihilated by $e(\mathbf{q})$, $\bar{e}(\mathbf{q})$, $a(\mathbf{q})$, $a_Q(\mathbf{q})$, $a_R(\mathbf{q})$, in which creation operators that commute with $a_Q(\mathbf{k})$ act on $|0\rangle$ — for example, $e^\dagger(\mathbf{q})|0\rangle$, $\bar{e}^\dagger(\mathbf{q})|0\rangle$, $a^\dagger(\mathbf{q})|0\rangle$, $a_Q^\star(\mathbf{q})|0\rangle$, but not $a_R^\star(\mathbf{q})|0\rangle$. Moreover, in the transformed ˆ representation, because Eq. (23) imposes Gauss's law on the transformed states, $|e(\mathbf{q})\rangle = e^\dagger(\mathbf{q})|0\rangle$ represents an electron *with* the electric and magnetic fields required to obey Gauss's law. The unitary transforms of the gauge fields are readily evaluated, but only the most interesting — $\hat{E}_l(\mathbf{x})$ and $\hat{B}(\mathbf{x})$ — will be given here. They are:

$$\hat{E}_l(\mathbf{x}) = E_l(\mathbf{x}) - \frac{1}{2\pi}\frac{\partial}{\partial x_l}\int d\mathbf{y}\ K_0(m|\mathbf{x}-\mathbf{y}|)j_0(\mathbf{y}) \tag{24}$$

and

$$\hat{B}(\mathbf{x}) = B(\mathbf{x}) - \frac{m}{2\pi}\int d\mathbf{y}\ K_0(m|\mathbf{x}-\mathbf{y}|)j_0(\mathbf{y}) \tag{25}$$

where $E_l(\mathbf{x})$ and $B(\mathbf{x})$ are the untransformed fields, and include only the contributions from the topologically massive photons and the ghost components of $\hat{E}_l(\mathbf{x})$ and $\hat{B}(\mathbf{x})$. They are:

$$
\begin{aligned}
E_l(\mathbf{x}) = &-\sum_{\mathbf{k}} \frac{imk_l}{k\sqrt{2\omega_k}}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\
&-\sum_{\mathbf{k}} \frac{\sqrt{\omega_k}\epsilon_{ln}k_n}{\sqrt{2}k}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\
&-\sum_{\mathbf{k}} \frac{8k^2\epsilon_{ln}k_n}{m^{5/2}}\left[a_Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a_Q^\star(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]
\end{aligned}
\tag{26}
$$

and

$$B(\mathbf{x}) = \sum_{\mathbf{k}} \frac{k}{\sqrt{2\omega_k}} \left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$
$$+ \sum_{\mathbf{k}} \frac{8k^3}{m^{5/2}} \left[a_Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a_Q^\star(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{27}$$

Equations (24) and (25) show that combinations of magnetic fields and longitudinal electric fields accompany charge densities in the transformed $\hat{}$ representation, implementing Gauss's law for the Fock space of charged states that solve Eq. (23). The transformed Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \hat{H}_{\text{quot}} + h + H_Q \tag{28}$$

where h has no dynamical significance because it is a total time derivative — it can be expresses as $h = i[H_0, \chi]$ or, equivalently, as $h = i[\hat{H}, \chi]$ where

$$\chi = -\sum_{\mathbf{k}} \frac{3m^{3/2}\phi(\mathbf{k})}{32k^3} j_0(\mathbf{k})j_0(-\mathbf{k}). \tag{29}$$

H_Q is complicated, but entirely composed of parts that are proportional to a_Q and $a_Q^\star$; thus, it plays *no role* in dynamical time-evolution of state vectors. $\hat{H}_{\text{quot}}$ operates within a quotient space of charged excitations (the 'electrons' and 'positrons') and the topologically massive photons. It has the form

$$\hat{H}_{\text{quot}} = H_{e\bar{e}} + \sum_{\mathbf{k}} \frac{\omega_k}{2} \left[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) \right] + \hat{H}_{\text{I}} \tag{30}$$

where $H_{e\bar{e}}$ 'counts' the topologically massive photons and electrons (positrons) and assigns them their appropriate energies. $\hat{H}_{\text{I}}$ is given by

$$\hat{H}_{\text{I}} = H_{\text{a}} + H_{\text{b}} + H_{j\gamma} \tag{31}$$

where H_{b} is an interaction between charge density and transverse current densities,

$$H_{\text{b}} = \int d\mathbf{x}\, d\mathbf{y}\; j_0(\mathbf{x})\epsilon_{ln}j_l(\mathbf{y})(x - y)_n \mathcal{F}(|\mathbf{x} - \mathbf{y}|) \tag{32}$$

and $\mathcal{F}(R)$ is a nonlocal interaction given by

$$\mathcal{F}(R) = -\frac{m}{2\pi} \int_0^\infty du\, \frac{J_1(u)}{u^2 + (mR)^2} \rightarrow \begin{cases} 1/4\pi R & mR \to 0 \\ 1/2\pi m R^2 & mR \to \infty \end{cases}. \tag{33}$$

H_{a} is a planar analog of the Coulomb interaction

$$H_{\text{a}} = \int d\mathbf{x}\, d\mathbf{y}\; j_0(\mathbf{x})j_0(\mathbf{y})K_0(m|\mathbf{x} - \mathbf{y}|) \tag{34}$$

where $K_0(m|\mathbf{x} - \mathbf{y}|)$ is the modified Bessel function; $K_0(u) \to \infty$ logarithmically when $u \to 0$. H_{a} is defined, however, for well-behaved $j_0(\mathbf{x})$ because the integration compensates

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for the singularity in $K_0(m|\mathbf{x} - \mathbf{y}|)$. $H_{j\gamma}$ is an interaction between topologically massive photons and currents, given by:

$$H_{j\gamma} = \sum_{\mathbf{k}} \frac{mk_l}{\sqrt{2}k\omega_k^{3/2}} \left[a(\mathbf{k})j_l(-\mathbf{k}) + a^\dagger(\mathbf{k})j_l(\mathbf{k}) \right]$$
$$- \sum_{\mathbf{k}} \frac{i\epsilon_{ln}k_n}{k\sqrt{2\omega_k}} \left[a(\mathbf{k})j_l(-\mathbf{k}) - a^\dagger(\mathbf{k})j_l(\mathbf{k}) \right]. \tag{35}$$

$\hat{H}$, which operates in a Fock space of states for which Gauss's law has been implemented, incorporates the feature that it *never* time-evolves states from the space of physical, propagating particles to the part of Hilbert space in which different varieties of ghosts coexist. The implementation of Gauss's law therefore naturally protects the unitarity of the theory. It is of special interest that $\hat{H}_{\text{quot}}$ is identical in *all* gauges (temporal, covariant, Coulomb, etc.) when Hamiltonians are unitarily transformed to the "standard" representation in which Eqs. (24) and (25) describe the relation between charges and the fields that surround them. Only the physically irrelevant H_Q depends on the gauge (for example, $H_Q = 0$ in the Coulomb gauge.)

IV. GAUGE-INVARIANCE AND GAUSS'S LAW

The Gauss's law operator $\mathcal{G}$ is the generator of gauge transformations:

$$e^{-i\int d\mathbf{y}\ \mathcal{G}(\mathbf{y})\chi(\mathbf{y})} \left\{ \begin{array}{c} A_i(\mathbf{x}) \\ \psi(\mathbf{x}) \end{array} \right\} e^{i\int d\mathbf{y}\ \mathcal{G}(\mathbf{y})\chi(\mathbf{y})} = \left\{ \begin{array}{c} A_i' = A_i + \partial_i\chi \\ \psi' = e^{ie\chi}\psi \end{array} \right\}. \tag{36}$$

In the 'transformed' ˆ representation,

$$\hat{\mathcal{G}} = \partial_l\hat{\Pi}_l - \frac{1}{4}m\epsilon_{ln}\hat{F}_{ln} + j_0 = \partial_l\Pi_l - \frac{1}{4}m\epsilon_{ln}F_{ln} \tag{37}$$

(j_0 is absent in the 'transformed' ˆ representation, although it is implicit in Π_l and A_l). $\hat{\mathcal{G}}$ manifestly commutes with ψ, so that ψ is a gauge-invariant spinor field in the 'transformed' ˆ representation:

$$\psi = \hat{\psi}_{\text{GI}} \tag{38}$$

What is $\hat{\psi}_{\text{GI}}$ in the original untransformed representation? We can use the U operator to transform *back* from the 'transformed' ˆ representation to the 'original' untransformed representation as shown by[1]

$$\psi_{\text{GI}} = U\hat{\psi}_{\text{GI}}U^{-1} = U\psi U^{-1} = e^{\mathcal{D}'(x)}\psi(x), \tag{39}$$

where

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[1]This corrects an error in the specification of the gauge-invariant spinor operators in our Refs. [2] and [3].

$$\mathcal{D}'(\mathbf{x}) = -ie \int d\mathbf{y} \ \Big\{ \xi_1'(|\mathbf{x} - \mathbf{y}|)\epsilon_{ln}\partial_l\Pi_n(\mathbf{y}) + \xi_2'(|\mathbf{x} - \mathbf{y}|)\partial_l A_l(\mathbf{y})$$
$$+ \ \eta_1'(|\mathbf{x} - \mathbf{y}|)\partial_l\Pi_l(\mathbf{x}) + \eta_2'(|\mathbf{x} - \mathbf{y}|)\epsilon_{ln}\partial_l A_n(\mathbf{x})$$
$$+ \ [\chi'(|\mathbf{x} - \mathbf{y}|) + \zeta'(|\mathbf{x} - \mathbf{y}|)]\,G(\mathbf{x}) \Big\} \tag{40}$$

and where

$$\xi_1'(|\mathbf{x} - \mathbf{y}|) = \sum_{\mathbf{k}} \frac{m}{\omega_k^2 k^2}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{41}$$

$$\xi_2'(|\mathbf{x} - \mathbf{y}|) = -\sum_{\mathbf{k}} \frac{1}{2}\left(\frac{1}{\omega_k^2} + \frac{1}{k^2} \right) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{42}$$

$$\eta_1'(|\mathbf{x} - \mathbf{y}|) = \sum_{\mathbf{k}} \frac{m^{3/2}\phi(\mathbf{k})}{8k^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{43}$$

$$\eta_2'(|\mathbf{x} - \mathbf{y}|) = \sum_{\mathbf{k}} \left(\frac{m^{5/2}\phi(\mathbf{k})}{16k^3} - \frac{1}{m\omega_k^2} \right) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{44}$$

$$\chi'(|\mathbf{x} - \mathbf{y}|) = \sum_{\mathbf{k}} \frac{1}{4k^2}\left[(1-\gamma) + \frac{2k^2}{m^2} - \frac{k^4}{m^2\omega^2} \right] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{45}$$

and

$$\zeta'(|\mathbf{x} - \mathbf{y}|) = -\sum_{\mathbf{k}} \frac{m^{3/2}\theta(\mathbf{k})}{8k^2}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \tag{46}$$

$\theta(\mathbf{k})$ and $\phi(\mathbf{k})$ are optional constituents of $\mathcal{D}'(\mathbf{x})$. They may be set to zero, or they may be any arbitrary real and even functions of $\mathbf{k}$.

In $\mathcal{D}'(x)$, if the gauge fields are gauge-transformed,

$$A_l \to A_l + \partial_l\chi, \tag{47}$$

$$\Pi_l \to \Pi_l + \frac{1}{2}m\epsilon_{ln}\partial_n\chi, \tag{48}$$

and, consequently,

$$\mathcal{D}'(x) \to \mathcal{D}'(x) - ie\chi \tag{49}$$

Since $\psi \to \psi e^{ie\chi}$, $e^{\mathcal{D}'(x)}\psi(x)$ is gauge-invariant. Also, $e^{\mathcal{D}'(x)}\psi(x)$ creates particles from the vacuum that obey Gauss's law.

$$[\mathcal{G}(y), e^{\mathcal{D}'(x)}\psi(x)] = 0. \tag{50}$$

This confirms our earlier demonstration that the charged particles, whose physical interactions are described by $\hat{H}_\mathrm{I}$, are gauge-invariant states and obey Gauss's law.

V. ROTATIONS, PARTICLE EXCHANGES, STATISTICS, ETC.

Statistical properties are unaffected by the implementation of Gauss's law:
Since unitary transformations do not change algebraic relations, such as equal-time commutation and anticommutation rules, we observe that

$$\{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \quad \{\psi(\mathbf{x}), \psi(\mathbf{y})\} = 0, \tag{51}$$

and also that,

$$\{\psi_{\mathrm{GI}}(\mathbf{x}), \psi^\dagger_{\mathrm{GI}}(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \quad \{\psi_{\mathrm{GI}}(\mathbf{x}), \psi_{\mathrm{GI}}(\mathbf{y})\} = 0. \tag{52}$$

The gauge-invariant 'electrons' and 'positrons' in the transformed ˆ representation therefore are anticommuting fermions that interact through the nonlocal interactions H_a and H_b, and through the photon-current interaction $H_{j\gamma}$.

Rotational phases can be arbitrary in this model:
Consider the rotation operator in the transformed ˆ representation. $\hat{J} = J + \mathcal{J}$ where

$$
J = -\int d\mathbf{x}\, \Pi_l x_i \epsilon_{ij} \partial_j A_l + \int d\mathbf{x}\, G x_i \epsilon_{ij} \partial_j A_0 - \int d\mathbf{x}\, \epsilon_{ij} \Pi_i A_j
$$
$$
- i \int d\mathbf{x}\, \psi^\dagger x_i \epsilon_{ij} \partial_j \psi - \frac{1}{2} \int d\mathbf{x}\, \psi^\dagger \gamma_0 \psi \tag{53}
$$

and

$$
\mathcal{J} = -\sum_{\mathbf{k}} \frac{m^{3/2}}{16 k^3} \epsilon_{ln} k_l \frac{\partial \phi(\mathbf{k})}{\partial k_n}\, j_0(-\mathbf{k}) j_0(\mathbf{k})
$$
$$
- \sum_{\mathbf{k}} i \epsilon_{ln} k_l \frac{\partial \theta(\mathbf{k})}{\partial k_n} \left[a_Q(\mathbf{k}) j_0(-\mathbf{k}) - a_Q^\star(\mathbf{k}) j_0(\mathbf{k}) \right]
$$
$$
+ \sum_{\mathbf{k}} \epsilon_{ln} k_l \frac{\partial \phi(\mathbf{k})}{\partial k_n} \left[a_Q(\mathbf{k}) j_0(-\mathbf{k}) + a_Q^\star(\mathbf{k}) j_0(\mathbf{k}) \right] \tag{54}
$$

If $\theta(k) = \phi(k) = 0$ (a perfectly consistent and viable choice), $\mathcal{J} = 0$, $\hat{J} = J$, and the charged states in the transformed ˆ representation rotate *precisely* as in the untransformed representation. But we can also choose, for example,

$$\phi(\mathbf{k}) = -\frac{8k^2}{m^{5/2}} \delta(k) \tan^{-1} \frac{k_2}{k_1}, \tag{55}$$

for which $\mathcal{J} = Q^2/4\pi m$ (Q is the charge operator) and, under a 2π rotation, an electron state in the transformed ˆ representation picks up the arbitrary "anyonic" phase $e^{i(e^2/4\pi m)}$. *This does not affect the statistical behavior, however.*

Particle exchange:
Consider a state $|A\rangle = e^\dagger(\mathbf{k}) e^\dagger(-\mathbf{k})|0\rangle$. If we use $e^{i(J+Q^2/4\pi m)\pi}$ to rotate $|A\rangle$ through π so that $\mathbf{k} \rightleftharpoons -\mathbf{k}$, we get

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$$|A'\rangle = -e^{ie^2/m}e^\dagger(-\mathbf{k})e^\dagger(\mathbf{k})|0\rangle \tag{56}$$

Suppose we interchange $e^\dagger(\mathbf{k})$ and $e^\dagger(-\mathbf{k})$ by commuting them. For the purposes of this discussion, we allow for the possibility of an 'exotic' graded commutator algebra:

$$e^\dagger(\mathbf{k})e^\dagger(-\mathbf{k}) + e^{i\delta}e^\dagger(-\mathbf{k})e^\dagger(\mathbf{k}) = 0. \tag{57}$$

If we exchange $e^\dagger(\mathbf{k})$ and $e^\dagger(-\mathbf{k})$ in $|A'\rangle$ with that commutator algebra, we get

$$|A''\rangle = e^{i(e^2/m+\delta)}e^\dagger(-\mathbf{k})e^\dagger(\mathbf{k})|0\rangle \tag{58}$$

Can we argue that the phase factor $e^{i(e^2/m+\delta)}$ must equal 1, so that the statistical properties of this model must match the arbitrary rotational phase? Our conclusion is that such an argument can not be supported.

In $3+1$ dimensions, the Lorentz group is so constraining that arbitrary rotational phases are not allowed. Vacuum expectation values of Lorentz-transformed fields are unique. Adding constant (like $Q^2/4\pi m$) to $\mathbf{J}$ in $3+1$ dimensions would violate $[J_i, J_j] = i\epsilon_{ijk}J_k$. The same feature that allows arbitrary rotational phases in $2+1$ dimensions, also affects the applicability, to this (2+1)-dimensional model, of the 'standard' proof of the spin-statistics connection. [6–8] One of the relatively few assumptions that underlie these proofs is the structure of the (complex) Lorentz group, which, in $3+1$ dimensions, uniquely associates a -1 factor with a 2π rotation of a Dirac spinor. That makes it necessary to reexamine the spin-statistics connection in spaces that allow arbitrary rotational phases.

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